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Wave propagation and Thom's theorem

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Abstract. Scalar waves propagated from a finite emitter in an inhomogeneous medium form caustics whose geometry and location are functions of both the boundary conditions on the emitter and the inhomogeneity of the medium. The wave amplitude satisfies a Helmholtz equation. For the class of problems considered below, a phase is derived from a characterisation of the medium and the boundary conditions on the emitter. The location and geometry of the caustic in configuration space is determined from phase space considerations through the Lagrange manifold. An oscillatory integral, whose asymptotic expansion is that of the amplitude, is determined by a sharpening of Maslov's method of characteristics. Thom's theorem prescribes a normal form for the phase function in the integral. Transformations carrying the phase function to the canonical form are determined; then, following Duistermaat, the complete asymptotic series of the integral, and hence of the amplitude, is obtained. The entire algorithm is illustrated with a specific example.

1. Introduction

Wave propagation in an inhomogeneous, non-dispersive medium is commonly represented by a partial differential equation of the form

$$\nabla_r^2\Psi(\boldsymbol{r}, t) = \frac{f(\boldsymbol{r})}{c^2}\frac{\partial^2\Psi(\boldsymbol{r}, t)}{\partial t^2} \tag{1}$$

where $\Psi(\boldsymbol{r}, t)$ is the wavefunction, $\boldsymbol{r}$ refers to the spatial coordinates, t is time, $f(\boldsymbol{r})$ is the profile characterising the inhomogeneity of the medium and c is the phase velocity when the medium is homogeneous, i.e. $f(\boldsymbol{r}) = 1$. For monochromatic waves transmitted at frequency ω with a harmonic time dependence, equation (1) becomes

$$\nabla_r^2\Psi(\boldsymbol{r}) + \tau^2 f(\boldsymbol{r})\Psi(\boldsymbol{r}) = 0 \tag{2}$$

where $\tau^2 = \omega^2/c^2$. Equation (2) is the reduced Helmholtz wave-equation, a second-order linear partial differential equation. Because no general technique exists for solving such equations, approximate solutions, valid under specific assumptions, are often constructed. One such approximation, valid at high frequencies, is the asymptotic series solution.

In the classical algorithm it is assumed that

$$\Psi(\boldsymbol{r}) \sim \exp(\mathrm{i}\tau\phi(\boldsymbol{r})) \sum_{k=0} a_k(\boldsymbol{r})\tau^{-k} \tag{3}$$

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as $\tau \to \infty$. Substitution of (3) into (2) leads to

$$(\nabla_r\phi)^2 - f(\boldsymbol{r}) = 0 \tag{4}$$

$$2(\nabla_r\phi)\cdot\nabla a_k + (\nabla_r^2\phi)a_k = -\nabla_r^2 a_{k-1}, \qquad k=0,1,2,\ldots, \qquad a_{-1}=0. \tag{5}$$

Equations (4) and (5) are the eikonal and transport equations, respectively. $\Psi(\boldsymbol{r})$ is determined by solving the eikonal for $\phi(\boldsymbol{r})$; then substituting $\phi(\boldsymbol{r})$ into the transport equation gives the $a_k(\boldsymbol{r})$ recursively. On caustic curves, i.e. the higher dimensional analogue of turning points, the $a_k(\boldsymbol{r})$ become unbounded. Then the procedure must be modified to remain valid (e.g. Zauderer 1970).

An alternative approach is a mixed configuration–momentum space formulation introduced by Maslov (1972) and extended by Arnold (1968, 1972a, b) and Duistermaat (1973, 1974). The field is represented by the integral

$$\Psi(\boldsymbol{r}) = \int A(\boldsymbol{r},\boldsymbol{p},\tau)\exp(\mathrm{i}\tau\phi(\boldsymbol{r},\boldsymbol{p}))\,\mathrm{d}\boldsymbol{p} \tag{6}$$

where

$$A(\boldsymbol{r},\boldsymbol{p},\tau) \sim \sum_k A_k(\boldsymbol{r},\boldsymbol{p})\tau^{-k}$$

and $\phi(\boldsymbol{r},\boldsymbol{p})$ has the form

$$\phi(\boldsymbol{r},\boldsymbol{p}) = \boldsymbol{r}\cdot\boldsymbol{p} - S(\boldsymbol{p})$$

where $S(\boldsymbol{p})$ is the generating function of a canonical transformation. $A(\boldsymbol{r},\boldsymbol{p},\tau)$ may be regarded as an amplitude and $\phi(\boldsymbol{r},\boldsymbol{p})$ as a phase, hence $\boldsymbol{p}$ is the wavevector, i.e. the normal to the surfaces of constant phase (wavefronts). Consequently (2) has an asymptotic solution of the form

$$\Psi(\boldsymbol{r}) - \int A(\boldsymbol{r},\boldsymbol{p},\tau)\exp[\mathrm{i}\tau(\boldsymbol{r}\cdot\boldsymbol{p} - S(\boldsymbol{p}))]\,\mathrm{d}\boldsymbol{p} = \mathrm{O}(\tau^{-\infty}). \tag{7}$$

Such integrals are evaluated using a stationary phase technique. At any field point, $\boldsymbol{r}_0$, the stationary phase condition ($\nabla_p\phi(\boldsymbol{r}_0,\boldsymbol{p})=0$) becomes

$$\boldsymbol{r}_0 = \nabla_p S(\boldsymbol{p}) \tag{8}$$

i.e. it defines a Lagrange manifold. A Lagrange manifold may be defined as a surface determined by the gradient of a generating function; here it is also seen as a transformation from momentum space to configuration space (Arnold 1978).

On the Lagrange manifold the eikonal condition (4) becomes $\boldsymbol{p}\cdot\boldsymbol{p} - f(\boldsymbol{r}) = 0$, from which Maslov defined his Hamiltonian

$$H = \boldsymbol{p}\cdot\boldsymbol{p} - f(\boldsymbol{r}). \tag{9}$$

Because the Lagrange manifold is invariant under the flow determined by (9), $S(\boldsymbol{p})$ and hence $\phi(\boldsymbol{r},\boldsymbol{p})$ may be determined from Hamilton–Jacobi theory (e.g. Berry 1976, Arnold 1978). Maslov's technique also determines the amplitude, $A(\boldsymbol{r},\boldsymbol{p},\tau)$, so that (7) holds with $\mathrm{O}(\tau^{-1})$, rather than $\mathrm{O}(\tau^{-\infty})$. A refinement of Maslov's technique determines an amplitude so that (7) holds as shown (Gorman and Wells 1981).

At regular stationary points, i.e. those points $\boldsymbol{r}_0$ at which the Hessian determinant of $\phi(\boldsymbol{r}_0,\boldsymbol{p})(\det(\partial^2\phi/\partial p_i\,\partial p_j))$ is non-zero, Kelvin's classical stationary phase technique (e.g. Bleistein and Handelsman 1975) suffices to determine the asymptotic series of the field integral in (7). At singular stationary points ($\det(\partial^2\phi/\partial p_i\,\partial p_j)=0$) where $\phi(\boldsymbol{r}_0, \boldsymbol{p})$ has finite codimension, the technique of Duistermaat (1973, 1974) obtains the asymptotic series of the field integral.

2. Synopsis

In this investigation we consider scalar waves, propagated from a finite source, in an inhomogeneous medium. Each field point will be specified by the range, i.e. the horizontal distance between two points on a plane, and the depth. For computational purposes the medium is often represented as a series of horizontal layers. Each stratum is separately characterised by a linear approximation to a depth-dependent profile valid only within the layer (Budden 1961, Brekhovskii 1962). Here, the medium is characterised by a single, continuous depth-dependent profile $f(x)$ which will be assumed invertible, i.e. $f^{-1}(x)$ is defined except at a finite number of isolated points. (At such points this algorithm does not apply.) Then the form of (2) we consider is

$$\nabla^2\psi(\boldsymbol{r})+\tau^2 f(x)\psi(\boldsymbol{r})=0 \tag{10}$$

where $\boldsymbol{r}=(x, y)$, x the depth and y the range. The phase will be so obtained as to enable an explicit representation of the emitter geometry and the boundary conditions on the wavevectors in $\phi(\boldsymbol{r}, \boldsymbol{p})$ itself. The determination of the caustic curve in configuration space proceeds from an analysis of the stationary points of $\phi(\boldsymbol{r}, \boldsymbol{p})$ in momentum space. A transport equation is derived in the mixed space representation which allows the determination of the higher order terms in the asymptotic series. The actual evaluation of the asymptotic series of the field integral (7) proceeds by transforming it to a form suitable for asymptotic analysis. This canonical form and the existence of the coordinate transformations required to obtain it follow from Thom's theorem (e.g. Brocker and Lander 1975, ch 15, Poston and Stewart 1978, ch 7); the explicit coordinate transformations are determined below. The complete asymptotic series both on and off the caustic is then determined. The entire algorithm is illustrated with a specific example.

3. Determination of the phase

We consider Maslov's Hamiltonian

$$H=p_x^2+p_y^2-f(x) \tag{11}$$

where $\boldsymbol{p}=(p_x, p_y)$ is the wavevector. We note here, however, that on the Lagrange manifold determined by $\boldsymbol{r}=\nabla_p S(\boldsymbol{p})$, the eikonal condition (4) becomes $\boldsymbol{p}\cdot\boldsymbol{p}-f(x)=0$. Consequently at any field point

$$x=f^{-1}(p_x^2+p_y^2) \tag{12}$$

determines one coordinate on the Lagrange manifold,

$$x=\frac{\partial S}{\partial p_x}. \tag{13}$$

Since

$$\frac{\partial^2 S}{\partial p_x \partial p_y} = \frac{\partial^2 S}{\partial p_y \partial p_x}$$

$$y = \int \frac{\partial^2 S(\boldsymbol{p})}{\partial p_x \partial p_y} \mathrm{d}p_x = \frac{\partial \tilde{S}(\boldsymbol{p})}{\partial p_y} + \theta(p_y) = \frac{\partial S(\boldsymbol{p})}{\partial p_y}. \tag{14}$$

$\tilde{S}(\boldsymbol{p})$ is a particular anti-derivative of $\partial\tilde{S}/\partial p_y$ (with respect to p_y), which in turn is an anti-derivative of the known function $\partial^2 S/\partial p_x \partial p_y$ with respect to p_x. $\theta(p_y)$ is an arbitrary function, introduced by the integration over p_x, the choice of which proceeds from boundary conditions on the wavevectors at the emitter. Equations (13) and (14) define the Lagrange manifold.

Let $\theta(p_y)$ be represented by a polynomial in p_y of arbitrary power with constant coefficients and let the geometry of the emitter be represented by

$$y = g(x). \tag{15}$$

The spatial coordinates of the emitter taken with the initial wavevectors may be regarded as a lifting from configuration space to the mixed configuration–momentum space. Then by choosing initial conditions on the emitter that satisfy (12), equations (13), (14) and (15) determine

$$g(x) = \frac{\partial \tilde{S}(\boldsymbol{p})}{\partial p_y} + a_0 + a_1 p_y + \ldots = \frac{\partial S(\boldsymbol{p})}{\partial p_y} \tag{16}$$

on the emitter. Successive differentiations of (16) with respect to x combined with (13) lead to a system of linear algebraic equations for the a_i in terms of the initial conditions of the wavevectors, thus determining $S(\boldsymbol{p})$ and hence the phase

$$\phi(\boldsymbol{r}, \boldsymbol{p}) = \phi(x, y, p_x, p_y) = xp_x + yp_y - S(p_x, p_y). \tag{17}$$

The equation of the caustic in configuration space proceeds from the Hessian of $\phi(\boldsymbol{r}, \boldsymbol{p})$. At caustic points the Hessian determinant vanishes (Berry 1976). Thus equating the Hessian determinant to zero determines the caustic curve in momentum space. Associated with each point, $\boldsymbol{p}_0$, on the caustic in momentum space is a point in configuration space determined by the Lagrange manifold, equations (13) and (14):

$$x_0 = \frac{\partial S(\boldsymbol{p}_0)}{\partial p_x}, \qquad y_0 = \frac{\partial S(\boldsymbol{p}_0)}{\partial p_y}.$$

The locus of these points determines the caustic in configuration space.

4. The transport equation

The transport equation is determined by carrying the differentiation (10) across the integral (7), yielding

$$\int \exp[\mathrm{i}\tau(\boldsymbol{r} \cdot \boldsymbol{p} - S(\boldsymbol{p}))][(\mathrm{i}\tau)^2(\boldsymbol{p} \cdot \boldsymbol{p} - f(x))A + 2\mathrm{i}\tau(\boldsymbol{p} \cdot \nabla_r A) + \nabla_r^2 A]\, \mathrm{d}\boldsymbol{p} = \mathrm{O}(\tau^{-\infty}).$$

The first term is Maslov's Hamiltonian (9) on the manifold. Expanding $\boldsymbol{p}\cdot\boldsymbol{p}-f(x)$,

$$\boldsymbol{p}\cdot\boldsymbol{p}-f(x)=\boldsymbol{p}\cdot\boldsymbol{p}-f\left(\frac{\partial S}{\partial p_x}\right)+(\boldsymbol{r}-\nabla_p S(\boldsymbol{p}))\cdot\boldsymbol{D}(\boldsymbol{r},\boldsymbol{p})=(\boldsymbol{r}-\nabla_p S(\boldsymbol{p}))\cdot\boldsymbol{D}(\boldsymbol{r},\boldsymbol{p}) \tag{18}$$

where the remainder term

$$D(\boldsymbol{r},\boldsymbol{p})=D=-\int_0^1 \nabla_r f(t(\boldsymbol{r}-\nabla_p S(\boldsymbol{p}))+\nabla_p S(\boldsymbol{p}))\,\mathrm{d}t.$$

Substituting into (18) leads to

$$\int \exp[\mathrm{i}\tau(\boldsymbol{r}\cdot\boldsymbol{p}-S(\boldsymbol{p}))]\mathrm{i}\tau\left(-D\cdot\nabla_p A-A\nabla_p\cdot\boldsymbol{D}+2\boldsymbol{p}\cdot\nabla_r A+\frac{1}{\mathrm{i}\tau}\nabla_r^2 A\right)\mathrm{d}\boldsymbol{p}=\mathrm{O}(\tau^{-\infty}). \tag{19}$$

For (7) to be an asymptotic solution, it is necessary that (19) be satisfied. A sufficient condition for (19) is that

$$-\boldsymbol{D}\quad\nabla_p A-A\nabla_p\cdot\boldsymbol{D}+2\boldsymbol{p}\cdot\nabla_r A+\frac{1}{\mathrm{i}\tau}\nabla_r^2 A=0 \tag{20}$$

in a neighbourhood of the Lagrange manifold. Then by introducing the flow

$$\begin{aligned}\dot{r}&=2\boldsymbol{p}\\ \dot{p}&=-\boldsymbol{D}(\boldsymbol{r},\boldsymbol{p})\end{aligned} \tag{21}$$

equation (20) leads to a transport equation in such a neighbourhood. That is, we allow the asymptotic series

$$A(\boldsymbol{r},\boldsymbol{p},\tau)\sim\sum_k A_k(\boldsymbol{r},\boldsymbol{p})\tau^{-k}$$

to evolve according to the transport equation

$$\dot{A}_k-A_k\nabla_p\cdot D+\nabla_r^2 A_{k-1}=0 \tag{22}$$

along the trajectories of (21) (Gorman and Wells 1981, Gorman *et al* 1980). (Note that in general the flow (21) is *not* the Hamiltonian flow.)

5. The coordinate transformations

The determination of the asymptotic series of the integrals

$$\int A_k(\boldsymbol{r},\boldsymbol{p})\exp\{\mathrm{i}\tau\phi(\boldsymbol{r},\boldsymbol{p})\}\,\mathrm{d}\boldsymbol{p}$$

(where $\phi(\boldsymbol{r},\boldsymbol{p})$ and $A_k(\boldsymbol{r},\boldsymbol{p})$ are obtained from equations (17) and (22) respectively) at any field point proceeds by transforming the phase to the form

$$\tilde{\phi}(\boldsymbol{r}_0,\boldsymbol{\beta})=\phi(\boldsymbol{r}_0,\boldsymbol{p}_0)\pm\beta_1^2\pm\beta_2^n. \tag{23}$$

Thom's theorem guarantees the existence of a coordinate transformation carrying $\phi(\boldsymbol{r},\boldsymbol{p})$ to the form (23) for Hessians of rank $\geqslant 1$. Off the caustic, the Hessian determinant of $\phi(\boldsymbol{r},\boldsymbol{p})$ is non-singular at $(\boldsymbol{r}_0,\boldsymbol{p}_0)$; then $n=2$ and the classical stationary phase technique applies to the transformed integral. On the caustic, the Hessian determinant is singular at $(\boldsymbol{r}_0, \boldsymbol{p}_0)$; then n is determined from the relative degeneracy of the Hessian based on a criterion derived from Thom's theorem (appendix). The classical stationary phase technique must then be modified to determine the asymptotic series. When $n = 2$, 3 or 4, explicit algebraic computation suffices to determine the required transformations. For $n > 4$ the determination of the appropriate coordinate transformation requires the algorithm specified in the proof of the splitting lemma (e.g. Gromoll and Meyer 1969, Poston and Stewart 1978, p 61), involving implicit rather than explicit solutions of equations. When $\phi(\boldsymbol{r}, \boldsymbol{p})$ is analytic the procedure can be made explicit, however, by an application of the Cauchy inversion theorem.

At $\boldsymbol{r}_0$ let $\phi(\boldsymbol{r}, \boldsymbol{p})$ have a stationary point $\boldsymbol{p}_0$, taken at $\boldsymbol{p}_0 = (0, 0)$ for clarity. Then the Taylor series of $\phi(\boldsymbol{r}, \boldsymbol{p})$ at $(\boldsymbol{r}_0, \boldsymbol{p}_0)$ is

$$\phi(\boldsymbol{r}_0, \boldsymbol{p}) = \phi(\boldsymbol{r}_0, \boldsymbol{0}) + C_1(p_x)p_x^2 + 2p_xp_yC_2(p_x, p_y) + C_3(p_y)p_y^n \tag{24}$$

where $C_1(0)$ and $C_3(0) \neq 0$. A linear (principal axis) transformation, followed by a regrouping, carries (24) to

$$\phi(\boldsymbol{r}_0, \xi_1, \xi_2) = \phi(\boldsymbol{r}_0, \boldsymbol{0}) + g_1(\xi_1, \xi_2)\xi_1^2 + 2\xi_1\xi_2^2h(\xi_2) + g_2(\xi_2)\xi_2^n \tag{25}$$

where $g_1(\boldsymbol{0})$, $g_2(0) \neq 0$. When $n = 2$, i.e. $g_1(\boldsymbol{0})$ and $g_2(\boldsymbol{0})$ are the eigenvalues, by completing the square we determine the coordinate transformation

$$\begin{aligned}\beta_1 &= (\tilde{g}_1)^{1/2}\xi_1 \pm (\tilde{g}_1)^{-1/2}h(\xi_2)\xi_2 \\ \beta_2 &= ((\tilde{g}_2) \mp (\tilde{g}_1)^{-1}h(\xi_2)^2)^{1/2}\xi_2\end{aligned} \tag{26}$$

where $\tilde{g}_i(\boldsymbol{0}) = g_i(\boldsymbol{0})$ when $g_i(\boldsymbol{0}) > 0$ and $\tilde{g}_i(\boldsymbol{0}) = -g_i(\boldsymbol{0})$ when $g_i(\boldsymbol{0}) < 0$, which carries $\phi(\boldsymbol{r}, \boldsymbol{p})$ at $(\boldsymbol{r}_0, \boldsymbol{p}_0)$ to

$$\tilde{\phi}(\boldsymbol{r}_0, \beta_1, \beta_2) = \phi(\boldsymbol{r}_0, \boldsymbol{0}) \pm \beta_1^2 \pm \beta_2^2 \tag{27}$$

where the signs of each β_i are determined by the sign of the corresponding eigenvalue. (The g_i may be computed from equations (24) and (25).) In (26) the positive sign in β_1 holds only when the Hessian at $(\boldsymbol{r}_0, \boldsymbol{p}_0)$ is positive definite; the positive sign in β_2 holds only when the Hessian is negative definite with $g_2(0)$ the positive eigenvalue. When $n = 3$, similarly, we determine the coordinate transformation

$$\begin{aligned}\beta_1 &= (\tilde{g}_1)^{1/2}\xi_1 \pm (\tilde{g}_1)^{-1/2}h(\xi_2)\xi_2^2 \\ \beta_2 &= (g_2(\xi_2) - g_1^{-1}h(\xi_2)^2\xi_2)^{1/3}\xi_2\end{aligned} \tag{28}$$

where the positive sign in β_1 holds only when $g_1(0) > 0$, which carries $\phi(\boldsymbol{r}, \boldsymbol{p})$ at $(\boldsymbol{r}_0, \boldsymbol{p}_0)$ to

$$\tilde{\phi}(\boldsymbol{r}_0, \beta_1, \beta_2) = \phi(\boldsymbol{r}_0, \boldsymbol{0}) \pm \beta_1^2 + \beta_2^3 \tag{29}$$

where the sign of β_1 is determined by the sign of $g_1(\boldsymbol{0})$. When $n = 4$ and $g_2(\boldsymbol{0}) - g_1(\boldsymbol{0})^{-1}h(0)^2 > 0$, the coordinate transformation

$$\begin{aligned}\beta_1 &= (\tilde{g}_1)^{1/2}\xi_1 \pm (\tilde{g}_1)^{-1/2}h(\xi_2)\xi_2^2 \\ \beta_2 &= (g_2(\xi_2) - g_1^{-1}h(\xi_2)^2)^{1/4}\xi_2^2\end{aligned} \tag{30}$$

carries $\phi(\boldsymbol{r}, \boldsymbol{p})$ at $(\boldsymbol{r}_0, \boldsymbol{p}_0)$ to

$$\tilde{\phi}(\boldsymbol{r}_0, \beta_1, \beta_2) = \phi(\boldsymbol{r}_0, \boldsymbol{0}) \pm \beta_1^2 + \beta_2^4; \tag{31}$$

when $n=4$ and $g_2(0)-g_1(\mathbf{0})^{-1}h(0)^2>0$, the coordinate transformation

$$\begin{aligned}\beta_1&=(\tilde{g}_1)^{1/2}\xi_1\pm(\tilde{g}_1)^{-1/2}h(\xi_2)\xi_2^2\\ \beta_2&=-(g_1^{-1}h(\xi_2)^2-g_2(\xi_2))^{1/4}\xi_2\end{aligned}\tag{32}$$

carries $\phi(\boldsymbol{r},\boldsymbol{p})$ at $(\boldsymbol{r}_0,\boldsymbol{p}_0)$ to

$$\tilde{\phi}(\boldsymbol{r}_0,\beta_1,\beta_2)=\phi(\boldsymbol{r}_0,\mathbf{0})\pm\beta_1^2-\beta_1^4.\tag{33}$$

In (30) and (32) the sign convention in β_1 is as in (28) and in (31) and (33) the sign of β_1 is determined by the sign of $g_1(\mathbf{0})$.

When $n>4$ explicit algebraic computation does not, in general, suffice to determine a coordinate transformation which carries $\phi(\boldsymbol{r},\boldsymbol{p})$ at $(\boldsymbol{r}_0,\boldsymbol{p}_0)$ to

$$\tilde{\phi}(\boldsymbol{r}_0,\beta_1,\beta_2)=\phi(\boldsymbol{r}_0,\mathbf{0})\pm\beta_j^2\pm\beta_k^n=\phi(\boldsymbol{r}_0,\boldsymbol{p}).\tag{34}$$

The procedure for determining the transformation can be made explicit however. Application of a principal axis transformation to $\phi(\boldsymbol{r},\boldsymbol{p})$ at $(\boldsymbol{r}_0,\boldsymbol{p}_0)$ determines

$$\Phi(\boldsymbol{r}_0,\xi_1,\xi_2)=\phi(\boldsymbol{r}_0,\mathbf{0})+g_1(\xi_1,\xi_2)\xi_1^2+2\xi_1\xi_2^2h(\xi_2)+l(\xi_2)\xi_2^n\tag{35}$$

where $g(\mathbf{0})$ is the non-vanishing eigenvalue and $l(\mathbf{0})\neq0$. Because $\partial_1^2\phi(\mathbf{0})\neq0$, $\partial_1^2\Phi(\mathbf{0})\neq0$; hence from the implicit function theorem, setting $\partial_1\phi(\xi_1,\xi_2)=0$ determines an implicit equation for ξ_1 in terms of ξ_2, e.g. $\xi_1=\theta(\xi_2)$. $\theta(\xi_2)$ may be determined explicitly, however, from the Cauchy inversion formula, namely

$$\theta(\xi_2)=\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{z\,\partial_1^2\Phi(z,\xi_2)\,\mathrm{d}z}{\partial_1\Phi(z,\xi_2)}\tag{36}$$

for fixed ξ_2, where γ is a circle in $\xi_2=0$ enclosing $\xi_1=0$. Then introducing the transformation

$$\xi_1=\alpha_1+\theta(\alpha_2)\qquad\xi_2=\alpha_2\tag{37}$$

equation (35) becomes

$$\begin{aligned}\Phi(\boldsymbol{r}_0,\alpha_1+\theta(\alpha_2),\alpha_2)&\\ =\phi(\boldsymbol{r}_0,\mathbf{0})&+g(\alpha_1+\theta(\alpha_2),\alpha_2)\alpha_1^2+2(\alpha_1+\theta(\alpha_2))\\ &\times\alpha_2^2h(\alpha_1+\theta(\alpha_2),\alpha_2)+l(\alpha_2)\alpha_2^n.\end{aligned}\tag{38}$$

Expanding (38) in a Taylor series about $\alpha_1=0$, with α_2 fixed, determines

$$\Phi(\boldsymbol{r}_0,\alpha_1+\theta(\alpha_2),\alpha_2)=\phi(\boldsymbol{r}_0,\mathbf{0})+\phi(\boldsymbol{r}_0,\theta(\alpha_2),\alpha_2)+\alpha_1^2R(\alpha_1,\alpha_2)\tag{39}$$

where $R(\alpha_1,\alpha_2)$ is the remainder of the series less a factor of α_1^2 ($R(\mathbf{0})\neq0$ because $\partial_1^2\phi(\mathbf{0})\neq0$) and where the coefficient of α_1 vanishes because $\partial_1\Phi(\xi_1,\xi_2)=0$. Then the transformation

$$\gamma_1=\alpha_1|R(\alpha_1,\alpha_2)|^{1/2}\qquad\gamma_2=\alpha_2$$

obtains

$$\hat{\phi}(\boldsymbol{r}_0,\gamma_1,\gamma_2)=\phi(\boldsymbol{r}_0,\mathbf{0})\pm\gamma_1^2+\mu(\gamma_2)=\Phi(\boldsymbol{r}_0,\alpha_1+\theta(\alpha_2),\alpha_2).\tag{40}$$

Since the first non-vanishing Taylor coefficient (at $\xi_2=0$) in the ξ_2 direction is the nth (equation (35)), $\mu(\gamma_2)=\gamma_2^n\tilde{\mu}(\gamma_2)$ with $\tilde{\mu}(0)\neq0$; then the transformation

$$\beta_1=\gamma_1\qquad\beta_2=\gamma_2|\tilde{\mu}(\gamma_2)|^{1/n}$$

carries $\hat{\phi}(\boldsymbol{r}_0, \gamma_1, \gamma_2)$ to

$$\tilde{\phi}(\boldsymbol{r}_0, \beta_1, \beta_2) = \phi(\boldsymbol{r}_0, \boldsymbol{0}) \pm \beta_1^2 \pm \beta_2^n \tag{41}$$

where the signs of the β_i must be determined for each specific case, as above. Equation (41) is the form required for the determination of the asymptotic series, cf equation (34).

Under the coordinate transformation which carries $\phi(\boldsymbol{r}, \boldsymbol{p})$ to the appropriate canonical form, the integral (6) at $(\boldsymbol{r}_0, \boldsymbol{p}_0)$ becomes

$$\int A(\boldsymbol{r}_0, \boldsymbol{p}, \tau) \exp(\mathrm{i}\tau\phi(\boldsymbol{r}, \boldsymbol{p}))\, \mathrm{d}\boldsymbol{p}$$

$$= \exp(\mathrm{i}\tau\phi(\boldsymbol{r}_0, \boldsymbol{p}_0)) \iint \tilde{A}(\boldsymbol{r}_0, \beta_1, \beta_2, \tau) \exp\{\mathrm{i}\tau(\pm\beta_1^2 \pm \beta_2^n)\}\, \mathrm{d}\beta_1\, \mathrm{d}\beta_2 \tag{42}$$

where

$$\tilde{A}(\boldsymbol{r}_0, \beta_1, \beta_2, \tau) = A(\boldsymbol{r}_0, p_x(\beta_1, \beta_2), p_y(\beta_1, \beta_2), \tau)\left(\frac{\partial(p_x, p_y)}{\partial(\beta_1, \beta_2)}\right).$$

$\tilde{A}(\boldsymbol{r}_0, \beta_1, \beta_2, \tau)$ may be determined by repeated applications of the Cauchy inversion theorem (cf Dieudonne 1960).

$$\tilde{A}(\boldsymbol{r}_0, \beta_1, \beta_2, \tau) = -\frac{1}{4\pi^2} \iint \frac{A(\boldsymbol{r}_0, \xi, \eta, \tau)}{(F_1(\xi, \eta) - \beta_1)(F_2(\xi, \eta) - \beta_2)} \frac{\partial(\beta_1, \beta_2)}{\partial(\xi, \eta)}\, \mathrm{d}\xi\, \mathrm{d}\eta$$

where

$$\beta_1 = F_1(\xi, \eta) \qquad \beta_2 = F_2(\xi, \eta).$$

6. Determination of the series

When $n = 2$ the asymptotic series of the transformed integral (42) is determined using the classical stationary phase technique, e.g. most elegantly, from Duistermaat's theorem

$$\iint \tilde{A}(\boldsymbol{r}_0, \beta_1, \beta_2) \exp[\mathrm{i}\tau(\pm\beta_1^2 \pm \beta_2^2)]\, \mathrm{d}\beta_1\, \mathrm{d}\beta_2 \sim \frac{2\pi}{\tau} \exp\left(\operatorname{sgn}\frac{\mathrm{i}\pi}{4}\right) \sum_{k=0} \frac{1}{k!} D^k A(\boldsymbol{r}_0, \boldsymbol{0}) \tau^{-k} \tag{43}$$

where

$$D = \frac{\mathrm{i}}{2}\left(\pm\frac{\partial^2}{\partial\beta_1^2} \pm \frac{\partial^2}{\partial\beta_2^2}\right)$$

and sgn is the number of positive eigenvalues less the number of negative eigenvalues and the sign of each derivative operator $\partial/\partial\beta_i$ is determined by the sign of β_i (Arnold 1972b). When $n \geq 3$ the classical stationary phase technique must be modified to remain valid. First, following Duistermaat, we express (42) as

$$\iint \tilde{A}(\boldsymbol{r}_0, \beta_1, \beta_2, \tau) \exp[\mathrm{i}\tau(\pm\beta_1^2 \pm \beta_2^n)]\, \mathrm{d}\beta_1\, \mathrm{d}\beta_2$$

$$= \int \exp(\pm \mathrm{i}\tau\beta_2^n)\, \mathrm{d}\beta_2 \int \tilde{A}(\boldsymbol{r}_0, \beta_1, \beta_2, \tau) \exp(\pm \mathrm{i}\tau\beta_1^2)\, \mathrm{d}\beta_1.$$

The asymptotic series of the integral over β_1 proceeds from Duistermaat's theorem, i.e.

$$\int \tilde{A}(\boldsymbol{r}_0, \beta_1, \beta_2, \tau) \exp(\pm \mathrm{i}\tau\beta_1^2)\, \mathrm{d}\beta_1 \sim \left(\frac{2\pi}{\tau}\right)^{1/2} \exp\left(\operatorname{sgn}\frac{\mathrm{i}\pi}{4}\right) \sum_k \frac{1}{k!}\left(\frac{\mathrm{i}}{2}\right)^k \boldsymbol{A}_{2k}(\boldsymbol{r}_0, \boldsymbol{0}, \beta_2)\tau^{-k} \tag{44}$$

where

$$\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2) = \frac{\partial^{2k}\tilde{A}(0, \beta_2, \tau)}{\partial\beta_1^{2k}}.$$

Each $\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2)$ determines an integral of the form

$$\int \boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2) \exp(\pm \mathrm{i}\tau\beta_2^n)\, \mathrm{d}\beta_2$$

which may be expressed as

$$I(\tau)\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2) = \int_{-\infty}^{\infty} \boldsymbol{A}(\boldsymbol{r}_0, 0, \beta_2) \exp(\pm \mathrm{i}\tau\beta_2^n)\, \mathrm{d}\beta_2 \tag{45}$$

owing to the boundedness of $A(\boldsymbol{r}, \boldsymbol{p}, \tau)$ and all its derivatives. Expanding $\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2)$ about $\beta_2 = 0$ determines

$$\begin{aligned}
&\int_{-\infty}^{\infty} \boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2) \exp(\pm \mathrm{i}\tau\beta_2^n)\, \mathrm{d}\beta_2 \\
&\qquad = \alpha_0 \int_{-\infty}^{\infty} \exp(\pm \mathrm{i}\tau\beta_2^n)\, \mathrm{d}\beta_2 + \alpha_1 \int_{-\infty}^{\infty} \exp(\pm \mathrm{i}\tau\beta_2^n)\beta_2\, \mathrm{d}\beta_2 + \ldots \\
&\qquad\quad + \alpha_{n-2} \int_{-\infty}^{\infty} \exp(\pm \mathrm{i}\tau\beta_2^n)\beta_2^{n-2}\, \mathrm{d}\beta_2 \\
&\qquad\quad + \frac{1}{n}\int_{-\infty}^{\infty} \exp(\pm \mathrm{i}\tau\beta_2^n)\beta_2^{n-1} \boldsymbol{R}\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2)\, \mathrm{d}\beta_2
\end{aligned} \tag{46}$$

where

$$\alpha_j = \left(\frac{1}{j!}\right) \frac{\mathrm{d}^j \boldsymbol{A}_{2k}(\boldsymbol{r}_0, \boldsymbol{0})}{\mathrm{d}\beta_2^j}$$

and

$$\boldsymbol{R}\boldsymbol{A}_{2k}(\boldsymbol{r}_0, \boldsymbol{0}, \beta_2) = n\beta_2^{1-n}\left[\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2) - \boldsymbol{A}_{2k}(\boldsymbol{r}_0, \boldsymbol{0}) - \sum_{j=1}^{n-2} \beta_2^j \left(\frac{1}{j!}\right) \frac{\mathrm{d}^j \boldsymbol{A}_{2k}(\boldsymbol{r}_0, \boldsymbol{0})}{\mathrm{d}\beta_2^j}\right]$$

i.e. the remainder of the Taylor series less a factor of n^{-1}.

The integrals

$$J_j(\tau) = \int_{-\infty}^{\infty} \exp(\pm \mathrm{i}\tau\beta_2^n)\beta_2^j\, \mathrm{d}\beta_2$$

are determined by contour integration. A partial integration of the last term in (46) gives

$$\begin{aligned}
&\frac{1}{n}\int_{-\infty}^{\infty} \exp(\pm \mathrm{i}\tau\beta_2^n)\beta_2^{n-1} \boldsymbol{R}\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2)\, \mathrm{d}\beta_2 \\
&\qquad = \mp\frac{1}{\mathrm{i}\tau}\int_{-\infty}^{\infty} \exp(\pm \mathrm{i}\tau\beta_2^n) \frac{\mathrm{d}}{\mathrm{d}\beta_2}(\boldsymbol{R}\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2))\, \mathrm{d}\beta_2
\end{aligned}$$

which follows from the boundedness of $A(\boldsymbol{r}, \boldsymbol{p}, \tau)$ and its derivatives (Guillemin and Sternberg 1977, pp 5–6). Let S be an operator defined by

$$\boldsymbol{S}\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2) = \frac{\mathrm{d}}{\mathrm{d}\beta_2}(\boldsymbol{R}\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2)) \qquad \text{i.e. } S = \frac{\mathrm{d}}{\mathrm{d}\beta_2} R.$$

Then (46) may be expressed as

$$I(\tau)\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2)$$

$$= \alpha_0 J_0(\tau) + \alpha_1 J_1(\tau) + \ldots + \alpha_{n-2} J_{n-2}(\tau) \mp \frac{1}{\mathrm{i}\tau} I(\tau) \boldsymbol{S}\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2) \tag{47a}$$

or alternatively

$$I(\tau)\left(\hat{1} \pm \frac{1}{\mathrm{i}\tau} S\right) = \hat{\alpha}_0 J_0(\tau) + \hat{\alpha}_1 J_1(\tau) + \ldots + \hat{\alpha}_{n-2} J_{n-2}(\tau) \tag{47b}$$

where $\hat{1}$ is the identity operator and the $\hat{\alpha}_k$ are operators which carry functions to operators.

$$I(\tau) = \left(\sum_{k=0}^{\infty} \hat{\alpha}_k J_k(\tau)\right)\left(\sum_{l=0}^{\infty} (\mp \mathrm{i}\tau)^{-l} S^l\right). \tag{48}$$

Therefore, equation (45) becomes

$$I(\tau)\boldsymbol{A}_{2k}(\boldsymbol{r}_0, 0, \beta_2) = \left(\sum_{j=0}^{n-2} \hat{\alpha}_j J_j(\tau)\right)\left[\sum_{l=0}^{\infty} (\mp \mathrm{i}\tau)^{-l} \left(\frac{\mathrm{d}}{\mathrm{d}\beta_2} R\right)^l \boldsymbol{A}_{2k}(\boldsymbol{r}_0, \boldsymbol{0})\right]. \tag{49}$$

Combining (44) and (49),

$$\iint \tilde{A}(\boldsymbol{r}_0, \beta_1, \beta_2, \tau) \exp[\mathrm{i}\tau(\pm\beta_1^2 \pm \beta_2^n)] \, \mathrm{d}\beta_1 \, \mathrm{d}\beta_2$$

$$\sim \left(\frac{2\pi}{\tau}\right)^{1/2} \exp\left(\pm \mathrm{i}\frac{\pi}{4}\right) \sum_{k=0} \frac{1}{k!}\left(\frac{\mathrm{i}}{2}\right)^k \tau^{-k} \left[\sum_{j=0}^{n-2} \hat{\alpha}_j J_j(\tau) \left(\sum_{l=0}^{\infty} (\mp \mathrm{i}\tau)^{-l}\right)\right.$$

$$\left. \times \left(\frac{\mathrm{d}}{\mathrm{d}\beta_2} R^l \boldsymbol{A}_{2k}(\boldsymbol{r}_0, \boldsymbol{o})\right)\right]. \tag{50}$$

For $n \geqslant 3$, equation (50) is the complete asymptotic series.

7. Example

To illustrate the algorithm we consider a medium characterised by a linear profile, $f(x) = x$. We investigate the far field, i.e. the distances involved are much larger than the dimensions of the emitter, produced by a line source, $y = 2x$, centred at $(5, -4)$. Let the magnitude of the momentum be $|\boldsymbol{p}| = 5$; for definiteness let the components of the initial momenta and the spatial variation of the momenta at the emitter be

$$\begin{array}{ll} p_{x0} = -2 & p_{y0} = 1 \\ p'_{x0} = 0.67 & p'_{y0} = -0.83 \\ p''_{x0} = 2.05 & p''_{y0} = 2.96 \\ p'''_{x0} = 10.2 & p'''_{y0} = 31.9 \end{array} \tag{51}$$

where the primes indicate differentiation with respect to x. At the emitter let $A_0(\boldsymbol{r}, \boldsymbol{p}) = 1$.

From (11) the Hamiltonian becomes

$$H = p_x^2 + p_y^2 - x = 0$$

leading to the Lagrange manifold, equations (13) and (14),

$$\begin{aligned} x &= p_x^2 + p_y^2 \\ y &= 2p_x p_y + 3 - 5p_y + p_y^2 + p_y^3. \end{aligned} \tag{52}$$

Therefore, for $(\boldsymbol{r}, \boldsymbol{p}) = (x, y, p_x, p_y)$

$$\phi(x, y, p_x, p_y) = xp_x + yp_y - \tfrac{1}{3}p_x^3 - p_x p_y^2 - 3p_y + \tfrac{5}{2}p_y^2 - \tfrac{1}{3}p_y^3 - \tfrac{1}{4}p_y^4 \tag{53}$$

from (17). By equating the determinant of the Hessian to zero we determine the equation of the caustic in momentum space,

$$4p_x^2 - 4p_y^2 - 10p_x + 4p_x p_y + 6p_x p_y^2 = 0. \tag{54}$$

Those real (p_x, p_y) satisfying (54) project the caustic onto configuration space through the Lagrange manifold (52) leading to the curve in figure 1.

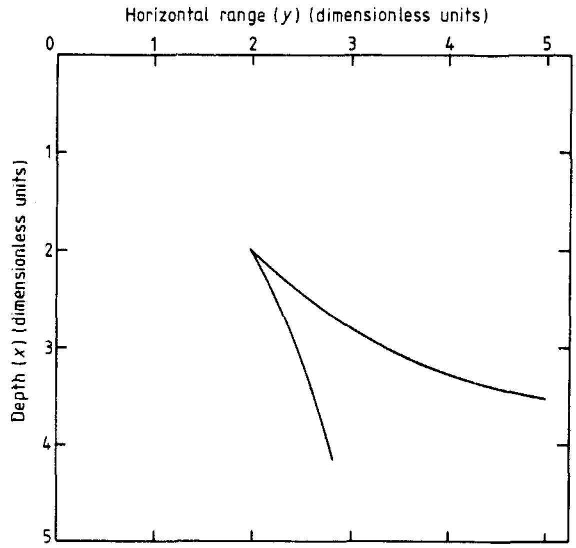

Figure 1. The caustic curve.

At the highest degenerate point on the caustic, $\boldsymbol{r} = (2, 2)$, $n = 4$; then the coordinate transformation

$$\begin{aligned} \beta_1 &= (1 - \tfrac{5}{8}\alpha_1 + \tfrac{1}{3}\alpha_2 + \tfrac{3}{32}\alpha_1^2 - \tfrac{1}{16}\alpha_1\alpha_2 + \tfrac{1}{64}\alpha_2^2)^{1/2}\alpha_2 \\ &\quad - (1 - \tfrac{5}{8}\alpha_1 + \tfrac{1}{3}\alpha_2 + \tfrac{3}{32}\alpha_1^2 - \tfrac{1}{16}\alpha_1\alpha_2 + \tfrac{1}{64}\alpha_2^2)^{-1/2}(\tfrac{1}{32}\alpha_1 - \tfrac{1}{4})\alpha_1^2 \\ \beta_2 &= [(\tfrac{1}{32}\alpha_1 - \tfrac{1}{4})^2(1 - \tfrac{5}{8}\alpha_1 + \tfrac{1}{3}\alpha_2 + \tfrac{3}{32}\alpha_1^2 - \tfrac{1}{16}\alpha_1\alpha_2 + \tfrac{1}{64}\alpha_2^2)^{-1} - \tfrac{1}{64}]^{1/4}\alpha_1 \end{aligned} \tag{55}$$

carries $\phi(\boldsymbol{r}, \boldsymbol{p})$ at (2, 2, 1, 1) to

$$\tilde{\phi}(2, 2, \beta_1, \beta_2) = \tfrac{19}{12} - \beta_1^2 + \beta_2^4.$$

Under the coordinate transformation (55) the field at $\boldsymbol{r} = (2, 2)$ is represented by

$$\psi(2,2) = \iint \tilde{A}(2, 2, \beta_1, \beta_2, \tau) \exp[\mathrm{i}\tau(\tfrac{19}{12} - \beta_1^2 + \beta_2^4)]\, \mathrm{d}\beta_1\, \mathrm{d}\beta_2. \tag{56}$$

The first two terms in the asymptotic series of (56) are then

$$\psi(2,2) \sim -0.77\tau^{-3/4} \exp(\tfrac{11}{6}\pi\mathrm{i})\Gamma(\tfrac{1}{4})(\cos\tfrac{1}{8}\pi + \mathrm{i}\sin\tfrac{1}{8}\pi)$$
$$-0.54\tau^{-5/4} \exp(\tfrac{11}{6}\pi\mathrm{i})\Gamma(\tfrac{3}{4})(\cos\tfrac{3}{8}\pi + \mathrm{i}\sin\tfrac{3}{8}\pi).$$

Equation (56) is the asymptotic solution for the emission defined by (51) on the line source $y = 2x$ centred at $(5, -4)$. The conditions given in (50) are a 4th order approximation to the solution of (51) at the emitter. It should be noted that if a higher order approximation to the solution of (51) were given, the location of the cusp would be unchanged; but the asymptotic series at the cusp would change slightly.

Appendix

Consider a function $\phi(\boldsymbol{r}, \boldsymbol{p})$, where $\boldsymbol{r} = (x, y)$ and $\boldsymbol{p} = (p_x, p_y)$ such that at $(\boldsymbol{r}_0, \boldsymbol{p}_0)$

$$\nabla_p \phi(\boldsymbol{r}_0, \boldsymbol{p}_0) = 0$$

and the Hessian $\partial^2\phi(\boldsymbol{r}_0, \boldsymbol{p}_0)/\partial p_i\, \partial p_j$ has at least one non-zero eigenvalue.

The normal form of $\phi(\boldsymbol{r}, \boldsymbol{p})$ at $(\boldsymbol{r}_0, \boldsymbol{p}_0)$ is then

$$\tilde{\phi}(\boldsymbol{r}_0, \boldsymbol{\beta}) = \phi(\boldsymbol{r}_0, \boldsymbol{p}_0) \pm \beta_1^2 \pm \beta_2^n$$

where the signs of the β_i are determined by the corresponding eigenvalues. If the Hessian determinant is non-zero at $(\boldsymbol{r}_0, \boldsymbol{p}_0)$, $n = 2$. If the Hessian determinant is zero at $(\boldsymbol{r}_0, \boldsymbol{p}_0)$ then

$$F(t) = \phi(\boldsymbol{r}_0, p_{x0} + te_1, p_{y0} + te_2)$$

where e_1, e_2 are the components of the eigenvector. The exponent of the first non-vanishing term in the Taylor series determines the value of n. The sign is determined by the sign of the Taylor coefficient.

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